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Regularity in codimension one of orbit closures in module varieties ☆

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Abstract

Let $\mathbb{M}_d(k)$ denote the space of $(d \times d)$-matrices with coefficients in an algebraically closed field k. Let X be an orbit closure in the product $[\mathbb{M}_d(k)]^t$ equipped with the action of the general linear group $\mathrm{GL}_d(k)$ by simultaneous conjugation. We show that X is regular at any point y such that the orbit of y has codimension one in X. The proof uses mainly the representation theory of associative algebras.

Keywords: Modules; Module varieties; Orbit closures

1. Introduction and the main results

Throughout the paper, k denotes an algebraically closed field and by an algebra we mean an associative k-algebra with an identity. Let d and t be positive integers. The points of $[\mathbb{M}_d(k)]^t$ correspond to the algebra homomorphisms from the free algebra $k\langle X_1, \ldots, X_t\rangle$ to $\mathbb{M}_d(k)$, or equivalently, to the left $k\langle X_1, \ldots, X_t\rangle$-modules with underlying vector space k^d. Furthermore, the isomorphism classes of d-dimensional left $k\langle X_1, \ldots, X_t\rangle$-

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modules correspond to the orbits in $[\mathbb{M}_d(k)]^t$ under the action of the general linear group $\mathrm{GL}_d(k)$ via

$$g \star (m_1, \dots, m_t) = \left(g m_1 g^{-1}, \dots, g m_t g^{-1}\right).$$

Now let A be a finitely generated algebra and $a_1, \dots, a_t$ be some generators, for a positive integer t. Then we get an isomorphism $A \simeq k\langle X_1, \dots, X_t\rangle/I$, where I is a two-sided ideal. Consequently, the set $\mathrm{mod}_A^d(k)$ of left A-modules with underlying vector space k^d can be identified with the $\mathrm{GL}_d(k)$-invariant closed subset of $[\mathbb{M}_d(k)]^t$ consisting of t-tuples $(m_1, \dots, m_t)$ such that $\rho(m_1, \dots, m_t)$ is the zero matrix for any (noncommutative) polynomial ρ in I. The affine variety $\mathrm{mod}_A^d(k)$ is called a module variety and depends on the choice of generators of A only up to a $\mathrm{GL}_d(k)$-equivariant isomorphism. We shall denote by $\mathcal{O}_M$ the $\mathrm{GL}_d(k)$-orbit of a module M in $\mathrm{mod}_A^d(k)$, and the closure of $\mathcal{O}_M$ with respect to the Zariski topology will be denoted by $\overline{\mathcal{O}}_M$. The main result of the paper solves the open problem posed by Bongartz in [5, §6.2, p. 598].

Theorem 1.1. *Let M and N be points in $\mathrm{mod}_A^d(k)$ such that N belongs to $\overline{\mathcal{O}}_M$ and $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 1$. Then the variety $\overline{\mathcal{O}}_M$ is regular at N.*

Let M be a module in $\mathrm{mod}_A^d(k)$, where A is a representation finite algebra, that is, there are only finitely many isomorphism classes of indecomposable modules in $\mathrm{mod}\, A$. Then $\overline{\mathcal{O}}_M$ contains only finitely many $\mathrm{GL}_d(k)$-orbits and hence we get the following result:

Corollary 1.2. *Let A be a representation finite algebra and d be a positive integer. Then the closures of $\mathrm{GL}_d(k)$-orbits in $\mathrm{mod}_A^d(k)$ are regular in codimension one.*

We also know that such orbit closures are unibranch, by [11], but we do not know if they are normal. The orbit closures in $\mathrm{mod}_A^d(k)$ are normal and Cohen–Macaulay provided A is the path algebra of a Dynkin quiver of type $\mathbb{A}_n$ or $\mathbb{D}_n$ [6], or A is a Brauer tree algebra [9]. We shall show in Section 2 that Theorem 1.1 follows from the following fact.

Theorem 1.3. *Let $0 \to Z \xrightarrow{\binom{\tilde f}{\tilde g}} Z \oplus Y \xrightarrow{(\tilde f, -\tilde h)} Z \to 0$ be a nonsplittable exact sequence of finite dimensional left A-modules with Z indecomposable. Then $\dim_k \mathrm{End}_A(Z) - \dim_k \mathrm{End}_A(Y) > 1$.*

We obtain from the above exact sequence two A-endomorphisms $x = \tilde g \tilde h$ and $y = \tilde g \tilde f \tilde h$ of the module Y. These endomorphisms satisfy the relations $xy = yx$ and $x^3 = y^2$, which allows to consider $\mathrm{End}_A(Y)$ as a bimodule over the ring $R = k[x, y]/(x^3 - y^2)$. Section 3 is devoted to the study of properties of modules and bimodules over the ring R related to the existence of their finite free resolutions. Results obtained there will be used in Section 4 to study the bimodule $\mathrm{End}_A(Y)$, leading to the proof of Theorem 1.3. Section 5 provides some consequences of Theorem 1.1 and additional remarks.

For basic background on the representation theory of algebras we refer to [2] and [8].

2. The proof of Theorem 1.1

Throughout the section, A is a finitely generated algebra and $\operatorname{mod} A$ denotes the category of finite dimensional left A-modules. Furthermore, we abbreviate $\dim_k \operatorname{Hom}_A(X, Y)$ to $[X, Y]$, for any modules X and Y in $\operatorname{mod} A$.

Lemma 2.1. *Let* $\sigma : 0 \to U \xrightarrow{f} W \xrightarrow{g} V \to 0$ *be an exact sequence in* $\operatorname{mod} A$ *and* X *be a module in* $\operatorname{mod} A$. *Then*

(1) $[U \oplus V, X] \geqslant [W, X]$ *and the equality holds if and only if any homomorphism in* $\operatorname{Hom}_A(U, X)$ *factors through* f;
(2) $[X, U \oplus V] \geqslant [X, W]$ *and the equality holds if and only if any homomorphism in* $\operatorname{Hom}_A(X, V)$ *factors through* g.

Proof. (1) follows from the induced exact sequence

$$0 \to \operatorname{Hom}_A(V, X) \xrightarrow{\operatorname{Hom}_A(g,X)} \operatorname{Hom}_A(W, X) \xrightarrow{\operatorname{Hom}_A(f,X)} \operatorname{Hom}_A(U, X)$$

and (2) follows by duality. □

Lemma 2.2. *Let* $\sigma : 0 \to U \xrightarrow{f} W \xrightarrow{g} V \to 0$ *be an exact sequence in* $\operatorname{mod} A$. *Then the following conditions are equivalent*:

(1) *the sequence* σ *splits*;
(2) W *is isomorphic to* $U \oplus V$;
(3) $[U \oplus V, U] = [W, U]$;
(4) $[V, U \oplus V] = [V, W]$.

Proof. Clearly the condition (1) implies (2), and the condition (2) implies (3) and (4). Applying Lemma 2.1 we get that (3) implies that the endomorphism 1_U factors through f, which means that f is a section and (1) holds. Similarly, it follows from (4) that g is a retraction and (1) holds. □

Throughout the section, M and N are two modules in $\operatorname{mod}_A^d(k)$ such that $N \in \overline{\mathcal{O}}_M$ and $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 1$. Applying [10, Theorem 1] we get modules Z, T and the exact sequences in $\operatorname{mod} A$

$$0 \to Z \xrightarrow{f} Z \oplus M \xrightarrow{g} N \to 0, \tag{2.1}$$

$$0 \to N \xrightarrow{f'} T \oplus M \xrightarrow{g'} T \to 0. \tag{2.2}$$

Lemma 2.3. $[M, M] = [M, N] = [N, M] = [N, N] - 1$.

Proof. Since the isotropy group of the point M can be identified with the automorphism group of the A-module M and the latter is open in the vector space $\mathrm{End}_A(M)$, then $\dim \mathcal{O}_M = \dim \mathrm{GL}_d(k) - [M, M]$. Similarly, $\dim \mathcal{O}_N = \dim \mathrm{GL}_d(k) - [N, N]$, which gives $[M, M] = [N, N] - 1$. Applying Lemmas 2.1 and 2.2 to the sequences (2.1) and (2.2) we get the inequalities

$$[M, M] \leqslant [N, M] < [N, N] \quad \text{and} \quad [M, M] \leqslant [M, N] < [N, N].$$

Now the claim follows easily. □

Let $\mathrm{rad}(-,-)$ denote the two-sided ideal of the functor

$$\mathrm{Hom}_A(-,-) : \mathrm{mod}\, A \times \mathrm{mod}\, A \to \mathrm{mod}\, k$$

generated by the nonisomorphisms between indecomposable modules. From now on, we assume that f belongs to $\mathrm{rad}(Z, Z \oplus M)$. In fact, if this is not the case, then f is of the form $\left(\begin{smallmatrix} f' & 0 \\ 0 & f'' \end{smallmatrix}\right) : Z' \oplus Z'' \to Z' \oplus (Z'' \oplus M)$, where f' is an isomorphism and f'' belongs to $\mathrm{rad}(Z'', Z'' \oplus M)$. Consequently, the exact sequence (2.1) has the form

$$0 \to Z' \oplus Z'' \xrightarrow{\left(\begin{smallmatrix} f' & 0 \\ 0 & f'' \end{smallmatrix}\right)} Z' \oplus (Z'' \oplus M) \xrightarrow{(0, g'')} N \to 0$$

and we can replace it by the exact sequence

$$0 \to Z'' \xrightarrow{f''} Z'' \oplus M \xrightarrow{g''} N \to 0.$$

Lemma 2.4. *There is an open neighbourhood $\mathcal{U}$ of f in $\mathrm{Hom}_A(Z, Z \oplus M)$ such that for any f' in $\mathcal{U}$ either f' is a section, or $f' = jfi$ for some A-endomorphisms i and j of Z and $Z \oplus M$, respectively.*

Proof. We first recall a construction described in [11] for the module $X = Z \oplus M$. Let $c = [X, M]$. The natural action of $\mathrm{GL}_d(k)$ on the space $\mathrm{Hom}_k(X, k^d)$ induces canonically an action of $\mathrm{GL}_d(k)$ on the Grassmann variety $\mathrm{Grass}(\mathrm{Hom}_k(X, k^d), c)$ of c-dimensional subspaces of the vector space $\mathrm{Hom}_k(X, k^d)$. We consider the $\mathrm{GL}_d(k)$-variety

$$\mathcal{C} = \mathrm{mod}_A^d(k) \times \mathrm{Grass}\big(\mathrm{Hom}_k\big(X, k^d\big), c\big),$$

and its one special $\mathrm{GL}_d(k)$-orbit

$$\mathcal{O}_{M_X} = \big\{\big(M', \mathrm{Hom}_A\big(X, M'\big)\big);\ M' \in \mathcal{O}_M\big\}.$$

Let $\pi : \overline{\mathcal{O}}_{M_X} \to \overline{\mathcal{O}}_M$ denote the restriction of the projection of $\mathcal{C}$ on $\mathrm{mod}_A^d(k)$. Now we want to construct a special regular morphism from an open subset of $\mathrm{Hom}_A(Z, X)$ to $\overline{\mathcal{O}}_{M_X}$ in a similar way as in the proof of [7, Proposition 3.4]. Let $e = \dim_k Z$. By choosing bases, we may assume that Z belongs to $\mathrm{mod}_A^e(k)$ and X belongs to $\mathrm{mod}_A^{e+d}(k)$. Then the elements of $\mathrm{Hom}_A(Z,X)$ can be considered as $((e+d)\times e)$-matrices. We choose an $((e+d)\times d)$-matrix b such that the matrix (f,b) is invertible. Observe that $\dim_k \mathrm{coker}(\mathrm{Hom}_A(X,f')) = c$ for any injective homomorphism $f': Z \to Z \oplus M$. Let $w_1,\dots,w_c$ be elements of $\mathrm{End}_A(X) \subseteq \mathbb{M}_{e+d}(k)$ whose residue classes form a basis of $\mathrm{coker}(\mathrm{Hom}_A(X,f))$. It is easy to see that there is an open neighbourhood $\mathcal{V}$ of f in $\mathrm{Hom}_A(Z,X)$ such that the matrix $[f',b]$ is invertible (in particular f' is injective) and the residue classes of $w_1,\dots,w_c$ form a basis of $\mathrm{coker}(\mathrm{Hom}_A(X,f'))$, for any homomorphism $f' \in \mathcal{V}$. Let $f' \in \mathcal{V}$, $g = [f',b]$ and write $g^{-1} = \left[\begin{smallmatrix} g' \\ g'' \end{smallmatrix}\right]$, where g' consists of the first e-rows of g^{-1}. Then $g^{-1} \star X = \left[\begin{smallmatrix} Z & W \\ 0 & N' \end{smallmatrix}\right]$, that is, N' is a module in $\mathrm{mod}_A^d(k)$ and

$$0 \to Z \xrightarrow{f'} X \xrightarrow{g''} N' \to 0$$

is an exact sequence in $\mathrm{mod}\, A$. We conclude from the induced exact sequence

$$0 \to \mathrm{Hom}_A(X,Z) \xrightarrow{\mathrm{Hom}_A(X,f')} \mathrm{Hom}_A(X,X) \xrightarrow{\mathrm{Hom}_A(X,g'')} \mathrm{Hom}_A\big(X,N'\big)$$

that $g''(w_1),\dots,g''(w_c)$ form a basis of the image $\mathrm{Im}(\mathrm{Hom}_A(X,g''))$. Hence we get a regular morphism $\Theta : \mathcal{V} \to \mathcal{C}$ sending f' to $(N', \mathrm{Im}(\mathrm{Hom}_A(X,g'')))$. If $f' \in \mathcal{V}$ is a section, then $N' \in \mathcal{O}_M$ and $\mathrm{Im}(\mathrm{Hom}_A(X,g'')) = \mathrm{Hom}_A(X,N')$, and consequently, $\Theta(f')$ belongs to the orbit $\mathcal{O}_{M_X}$. Since the sections in $\mathcal{V}$ form an open subset of the irreducible set $\mathcal{V}$, then the image of Θ is contained in $\overline{\mathcal{O}}_{M_X}$. On the other hand, if $f' \in \mathcal{V}$ is not a section, then N' is not isomorphic to M, which implies that $\Theta(f')$ belongs to the boundary $\partial\mathcal{O}_{M_X} = \overline{\mathcal{O}}_{M_X} \setminus \mathcal{O}_{M_X}$ of $\mathcal{O}_{M_X}$. Since $\dim \partial\mathcal{O}_{M_X} < \dim \mathcal{O}_{M_X} = \dim \mathcal{O}_M = \dim \mathcal{O}_N + 1$, the inverse image $\pi^{-1}(\mathcal{O}_N)$ is a finite (disjoint) union of $\mathrm{GL}_d(k)$-orbits and each of them is open in $\partial\mathcal{O}_{M_X}$. Let $\mathcal{O}_1$ denote the one containing $\Theta(f)$. Then $\mathcal{O}_{M_X} \cup \mathcal{O}_1$ is an open subset of $\overline{\mathcal{O}}_{M_X}$, and consequently, $\mathcal{U} = \Theta^{-1}(\mathcal{O}_{M_X} \cup \mathcal{O}_1)$ is an open subset of $\mathcal{V}$.

Assume that $\Theta(f') = (N', \mathrm{Im}(\mathrm{Hom}_A(X,g'')))$ belongs to $\mathcal{O}_1$. Then $N' = h \star N$ and $\mathrm{Im}(\mathrm{Hom}_A(X,g'')) = h \star \mathrm{Im}(\mathrm{Hom}_A(X,g))$ for some element h in $\mathrm{GL}_d(k)$. Hence $h : N \to N'$ is an A-isomorphism and $\mathrm{Im}(\mathrm{Hom}_A(X,g'')) = \mathrm{Im}(\mathrm{Hom}_A(X,hg))$. In particular, $hg = g''j$ for some $j \in \mathrm{End}_A(X)$. Thus we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "f"] \arrow[d, "i'"'] & X \arrow[r, "g"] \arrow[d, "j"] & N \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & Z \arrow[r, "f'"] & X \arrow[r, "g''"] & N' \arrow[r] & 0.
\end{tikzcd}$$

Since h is an isomorphism, the sequence

$$0 \to Z \xrightarrow{\binom{f}{i'}} X \oplus Z \xrightarrow{(j,-f')} X \to 0$$

is exact. Then the homomorphism $\binom{f}{i'}$ is a section, by Lemma 2.2. The same is true for the endomorphism i', as f belongs to $\mathrm{rad}(Z, X)$. Hence i' is an isomorphism and $f' = jfi$, where $i = (i')^{-1}$. $\square$

Let $\mathrm{rad}_A(X)$ denote the Jacobson radical of a module X in $\mathrm{mod}\, A$ and let $\mathrm{rad}(E)$ denote the Jacobson radical of an algebra E. Then $\mathrm{rad}(\mathrm{End}_A(X)) = \mathrm{rad}(X, X)$ for any module X in $\mathrm{mod}\, A$. Moreover, if X is indecomposable, then the algebra $\mathrm{End}_A(X)$ is local with the maximal ideal $\mathrm{rad}(X, X)$ consisting of the nilpotent endomorphisms of X and there is a decomposition $\mathrm{End}_A(X) = k \cdot 1_X \oplus \mathrm{rad}(X, X)$.

Lemma 2.5. *The module Z is indecomposable, $[N, Z] = [M, Z] + 1$ and any radical endomorphism of Z factors through f.*

Proof. Suppose that $Z = Z_1 \oplus Z_2$ for two nonzero modules Z_1 and Z_2. Since f belongs to $\mathrm{rad}(Z_1 \oplus Z_2, Z_1 \oplus Z_2 \oplus M)$, then the map $f + t \cdot 1_{Z_1} : Z \to Z \oplus M$ is not a section, for any $t \in k$. Applying Lemma 2.4, we get $f + t \cdot 1_{Z_1} = jfi$ for some $t \neq 0$ and endomorphisms i and j. Since f belongs to $\mathrm{rad}(Z, Z \oplus M)$, the same holds for $f + t \cdot 1_{Z_1}$ and $t \cdot 1_{Z_1}$, a contradiction. Therefore the module Z is indecomposable.

Let $E = \mathrm{End}_A(Z)$. We have the induced exact sequence in $\mathrm{mod}\, E$

$$0 \to \mathrm{Hom}_A(N, Z) \xrightarrow{\mathrm{Hom}_A(g,Z)} \mathrm{Hom}_A(Z \oplus M, Z) \xrightarrow{\mathrm{Hom}_A(f,Z)} \mathrm{Hom}_A(Z, Z).$$

Then the image of $\alpha = \mathrm{Hom}_A(f, Z)$ is contained in $\mathrm{rad}(E) = \mathrm{rad}(Z, Z)$ as f belongs to $\mathrm{rad}(Z, Z \oplus M)$. It remains to show the reverse inclusion, which means that the restriction

$$\alpha' : \mathrm{Hom}_A(Z \oplus M, Z) \to \mathrm{rad}(E)$$

of α is surjective. Since the image of α' is an E-submodule and $\mathrm{rad}_E(\mathrm{rad}(E)) = \mathrm{rad}^2(E)$, it suffices to show that the composition

$$\beta : \mathrm{Hom}_A(Z \oplus M, Z) \to \mathrm{rad}(E)/\mathrm{rad}^2(E)$$

of α' followed by a quotient is surjective.

Let $h \in \mathrm{rad}(E)$. Observe that $f + t \cdot \binom{h}{0}$ belongs to $\mathrm{rad}(Z, Z \oplus M)$ for any $t \in k$. Applying Lemma 2.4 we get $f + t \cdot \binom{h}{0} = jfi$ for some $t \neq 0$ and endomorphisms i and j. Then we have the equality $(1_Z, 0)f + t \cdot h = j'fi$ in E, where $j' = (1_Z, 0)j : Z \oplus M \to Z$. We decompose $i = c \cdot 1_Z + i'$, where $c \in k$ and $i' \in \mathrm{rad}(E)$. Since $j'f$ belongs to $\mathrm{rad}(E)$, then $j'fi - cj'f$ belongs to $\mathrm{rad}^2(E)$. Altogether, we conclude that

$$h + \mathrm{rad}^2(E) = t^{-1} \cdot \big(cj' - (1_Z, 0)\big)f + \mathrm{rad}^2(E) = \beta\big(ct^{-1}j' - \big(t^{-1} \cdot 1_Z, 0\big)\big),$$

which finishes the proof. $\square$

We decompose $f = \binom{f_1}{f_2}$ and $g = (g_1, g_2)$. Then the square

$$\begin{array}{ccc} Z & \xrightarrow{f_2} & M \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle -g_2} \\ Z & \xrightarrow{g_1} & N \end{array} \tag{2.3}$$

is exact, that is, it is a pushout and a pull-back. Furthermore f_1 is nilpotent.

Lemma 2.6. *Let j be a positive integer such that $(f_1)^j = 0$. Then any radical endomorphism of Z factors through $(f_1, b): Z \oplus M^j \to Z$ for some A-homomorphism b.*

Proof. Let e' be an element of $\mathrm{End}_A(Z)$. Applying Lemma 2.5 we obtain a decomposition

$$e' = \lambda \cdot 1_Z + a' f_1 + b' f_2$$

for some scalar $\lambda \in k$ and A-homomorphisms $a': Z \to Z$ and $b': M \to Z$. Let e be a radical endomorphism of Z. Using the above j times, we get

$$e = \sum_{i=0}^{j-1} \lambda_i \cdot (f_1)^i + b_i f_2 (f_1)^i.$$

Since the endomorphism e is radical then $\lambda_0 = 0$. Hence we get the claim for $b = (b_0, \dots, b_{j-1})$. $\square$

Proposition 2.7. *Assume that Theorem* 1.3 *holds. Then* $[Z, M] = [Z, N]$.

Proof of Proposition 2.7. Suppose that $[Z, M] \neq [Z, N]$. We divide the proof into several steps.

Step 1. *g_1 factors through $f = \binom{f_1}{f_2}$.*

Proof. Applying Lemma 2.1 for $X = Z$ and the sequence (2.1) we get a homomorphism u in $\mathrm{Hom}_A(Z, N)$ which does not factor through g. Furthermore, we may assume that uf_1 factors through g as f_1 is nilpotent. Hence

$$uf_1 = g_1 a_1 + g_2 a_2 \tag{2.4}$$

for some A-homomorphisms $a_1: Z \to Z$ and $a_2: Z \to M$. The homomorphism a_2 factors through f, by Lemmas 2.3 and 2.1 applied for $X = M$ and the sequence (2.1). We decompose the endomorphism $a_1 = \lambda \cdot 1_Z + a_1'$, where $\lambda \in k$ and a_1' belongs to $\mathrm{rad}(Z, Z)$. By Lemma 2.5, a_1' also factors through f, and consequently,

$$a_1 = \lambda \cdot 1_Z + b_{1,1} f_1 + b_{1,2} f_2, \qquad a_2 = b_{2,1} f_1 + b_{2,2} f_2$$

for some A-homomorphisms $b_{1,1}$, $b_{1,2}$, $b_{2,1}$ and $b_{2,2}$. Combining these equalities with (2.4) we get

$$\lambda \cdot g_1 = (u - g_1 b_{1,1} - g_2 b_{2,1}) f_1 + (-g_1 b_{1,2} - g_2 b_{2,2}) f_2. \tag{2.5}$$

Suppose that $\lambda = 0$. Then it follows from the exactness of (2.3) that

$$u - g_1 b_{1,1} - g_2 b_{2,1} = c g_1$$

for some A-endomorphism $c : N \to N$. We know that $[N, N] - [N, M] = 1$, by Lemma 2.3. We conclude from the induced exact sequence

$$0 \to \mathrm{Hom}_A(N, Z) \xrightarrow{\mathrm{Hom}_A(N, f)} \mathrm{Hom}_A(N, Z \oplus M) \xrightarrow{\mathrm{Hom}_A(N, g)} \mathrm{Hom}_A(N, N)$$

that $\mathrm{Hom}_A(N, N) = k \cdot 1_N \oplus \mathrm{Im}(\mathrm{Hom}_A(N, g))$. Hence $c = \mu \cdot 1_N + g_1 d_1 + g_2 d_2$ for some $\mu \in k$ and A-homomorphisms d_1 and d_2. Consequently, the homomorphism

$$u = g_1(b_{1,1} + \mu \cdot 1_Z + d_1 g_1) + g_2(b_{2,1} + d_2 g_1)$$

factors through $g = (g_1, g_2)$, a contradiction. Thus $\lambda \neq 0$ and the equality (2.5) shows that g_1 factors through $f = \binom{f_1}{f_2}$. $\square$

Hence the exact square (2.3) divides into two exact squares

$$\begin{tikzcd} Z \arrow[r, "u"] \arrow[d, "f_1"'] & W \arrow[r, "x"] \arrow[d, "\binom{w_1}{w_2}"] & M \arrow[d, "-g_2"] \\ Z \arrow[r, "\binom{f_1}{f_2}"] & Z \oplus M \arrow[r, "(y_1, y_2)"] & N. \end{tikzcd} \tag{2.6}$$

Step 2. *The homomorphism $w_2 : W \to M$ is a retraction and the inequality $[Z, Z \oplus M] - [Z, W] \leqslant 1$ holds.*

Proof. Applying Lemma 2.1 for $X = M$ and the exact squares (2.6), we get that the integers

$$[M, Z \oplus M] - [M, W] \quad \text{and} \quad [M, W \oplus N] - \big[M, M^2 \oplus Z\big]$$

are nonnegative. By Lemma 2.3, their sum equals $[M, N] - [M, M] = 0$. Hence these numbers are zero and any map in $\mathrm{Hom}_A(M, Z \oplus M)$ factors through $\begin{pmatrix} f_1 & w_1 \\ f_2 & w_2 \end{pmatrix}$, by Lemma 2.1 applied for $X = M$ and the left square in (2.6). Consequently, any map in $\mathrm{Hom}_A(M, Z)$ factors through (f_1, w_1) while any endomorphism in $\mathrm{End}_A(M)$ factors through (f_2, w_2). In particular, (f_2, w_2) is a retraction and the same holds for w_2, as f_2 belongs to $\mathrm{rad}(Z, M)$.

Furthermore, $\mathrm{Hom}_A(Z, M)$ is contained in the image of the map $\alpha = \mathrm{Hom}_A\big(Z, \left(\begin{smallmatrix} f_1 & w_1 \\ f_2 & w_2 \end{smallmatrix}\right)\big)$ in the induced exact sequence

$$0 \to \mathrm{Hom}_A(Z, Z) \to \mathrm{Hom}_A(Z, Z \oplus W) \xrightarrow{\alpha} \mathrm{Hom}_A(Z, Z \oplus M).$$

Hence the inequality $[Z, Z \oplus M] - [Z, W] \leqslant 1$ will be a consequence of the fact that $\mathrm{rad}(Z, Z)$ is contained in the image of the map

$$\mathrm{Hom}_A\big(Z, (f_1, w_1)\big) : \mathrm{Hom}_A(Z, Z \oplus W) \to \mathrm{Hom}_A(Z, Z).$$

The latter follows from Lemma 2.6 and the fact that any homomorphism in $\mathrm{Hom}_A(M, Z \oplus M)$ factors through (f_1, w_1). $\square$

Consequently, we can decompose $W = Y \oplus M$ for some A-module Y, such that $w_2 = (0, 1_M) : Y \oplus M \to M$.

Step 3. *There is a nonsplittable exact sequence in* $\mathrm{mod}\, A$ *of the form*

$$0 \to Z \xrightarrow{\left(\begin{smallmatrix} \tilde{f} \\ \tilde{g} \end{smallmatrix}\right)} Z \oplus Y \xrightarrow{(\tilde{f}, -\tilde{h})} Z \to 0. \tag{2.7}$$

Proof. We decompose $u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) : Z \to Y \oplus M$ and $w_1 = (v_1, v_2) : Y \oplus M \to Z$. We conclude from (2.6) the exactness of the upper row in the diagram

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & Z \arrow[r, "\left(\begin{smallmatrix} f_1 \\ u_1 \\ u_2 \end{smallmatrix}\right)"] \arrow[d, equal] & Z \oplus Y \oplus M \arrow[r, "\left(\begin{smallmatrix} f_1 & -v_1 & -v_2 \\ f_2 & 0 & -1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -f_2 & 0 & 1 \end{smallmatrix}\right)"'] & Z \oplus M \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 & -v_2 \\ 0 & -1 \end{smallmatrix}\right)"] & 0 \\
0 \arrow[r] & Z \arrow[r, "\left(\begin{smallmatrix} f_1 \\ u_1 \\ 0 \end{smallmatrix}\right)"] & Z \oplus Y \oplus M \arrow[r, "\left(\begin{smallmatrix} f_1 - v_2 f_2 & -v_1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] & Z \oplus M \arrow[r] & 0
\end{tikzcd} \tag{2.8}$$

In particular, $f_2 f_1 - u_2 = 0$, which implies that the diagram (2.8) is commutative. Since the maps corresponding to vertical arrows are isomorphisms, the bottom row is exact as well. It follows from the construction of the squares (2.6) that $f_2 = xu$. We decompose $x = (x_1, x_2) : Y \oplus M \to M$. Then

$$f_2 = x_1 u_1 + x_2 u_2 = x_1 u_1 + x_2 f_2 f_1,$$

and consequently,

$$f_1 - v_2 f_2 = (1_Z - v_2 x_2 f_2) f_1 - v_2 x_1 u_1.$$

Since f_2 belongs to $\operatorname{rad}(Z, M)$, the endomorphism $a = 1_Z - v_2 x_2 f_2$ is an isomorphism. Then

$$f_1 - v_2 f_2 = a f_1 + a b u_1,$$

where $b = -a^{-1} v_2 x_1$. The exactness of the bottom row in the diagram (2.8) implies the exactness of the upper row in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "\binom{f_1}{u_1}"] \arrow[d, equal] & Z \oplus Y \arrow[r, "(af_1+abu_1, -v_1)"] \arrow[d, "\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)"'] & Z \arrow[r] \arrow[d, "(a^{-1})"] & 0 \\
0 \arrow[r] & Z \arrow[r, "\binom{f_1+bu_1}{u_1}"] & Z \oplus Y \arrow[r, "(f_1+bu_1, -\tilde{h})"] & Z \arrow[r] & 0
\end{tikzcd}$$

where $\tilde{h} = a^{-1} v_1 + f_1 b + b u_1 b$. Since the maps corresponding to the vertical arrows are isomorphisms, the bottom row is also exact. Setting $\tilde{f} = f_1 + b u_1$ and $\tilde{g} = u_1$ we get the exact sequence (2.7). Suppose that the sequence (2.7) splits. Since $\tilde{f} = a^{-1} f_1 - a^{-1} v_2 f_2$ belongs to $\operatorname{rad}(Z, Z)$, then $\tilde{g} : Z \to Y$ is a section and $\tilde{h}$ is a retraction. Hence both of them are isomorphisms as $\dim_k Y = \dim_k Z$. Consequently, $\tilde{f}\tilde{f} = \tilde{h}\tilde{g}$ is an isomorphism, a contradiction. Therefore the exact sequence (2.7) does not split. $\square$

The right square in (2.6) leads to the exact sequence

$$0 \to Y \oplus M \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 \\ v_1 & v_2 \\ 0 & 1 \end{smallmatrix}\right)} M \oplus Z \oplus M \xrightarrow{(g_2, y_1, y_2)} N \to 0,$$

which implies the exactness of the sequence

$$0 \to Y \xrightarrow{\binom{x_1}{v_1}} M \oplus Z \xrightarrow{(g_2, y_1)} N \to 0. \tag{2.9}$$

Step 4. $[Z, Z] - [Y, Y] = 1$.

Proof. We claim that $[Z, Y] = [Y, Y]$. Assume first that the exact sequence (2.9) splits. Since the sequence (2.7) does not split, then Y is not isomorphic to Z. Hence v_1 belongs to $\operatorname{rad}(Y, Z)$, as Z is indecomposable and $\dim_k Z = \dim_k Y$. This implies that $x_1 : Y \to M$ is a section. In particular, M is isomorphic to $Y \oplus Y'$ for some A-module Y'. Applying Lemma 2.1 to (2.7) we get $[Y, Y] \leqslant [Z, Y]$ and $[Y, Y'] \leqslant [Z, Y']$, and applying it to (2.9) we get $[M \oplus Z, M] \leqslant [Y \oplus N, M]$. Consequently,

$$0 \leqslant [Z, Y] - [Y, Y] \leqslant [Z, M] - [Y, M] \leqslant [N, M] - [M, M] = 0,$$

by Lemma 2.3.

Assume now that the exact sequence (2.9) does not split. Then

$$[M, Y \oplus N] \geqslant [M, M \oplus Z], \qquad [N, Z] \geqslant [N, Y],$$

$$[Y \oplus N, Y] > [M \oplus Z, Y], \qquad [Z, Y] \geqslant [Y, Y],$$

by Lemmas 2.1 and 2.2 applied to the sequences (2.7) and (2.9). From Lemmas 2.3 and 2.5 we get $[M, M] = [M, N]$, $[N, Z] - [M, Z] = 1$ and hence

$$\begin{aligned} 0 &\leqslant [Z, Y] - [Y, Y] \leqslant [N, Y] - [M, Y] - 1 \\ &\leqslant \big([N, Z] - [M, Z] - 1\big) + \big([M, Z] - [M, Y]\big) \leqslant [M, N] - [M, M] = 0, \end{aligned}$$

which proves the claim.

By Step 2, we get $[Z, Z] - [Z, Y] \leqslant 1$. But $[Z, Z] > [Z, Y]$, by Step 3 and Lemmas 2.1 and 2.2. Therefore $[Z, Z] - [Y, Y] = [Z, Z] - [Z, Y] = 1$. □

Steps 3 and 4 give a contradiction with Theorem 1.3. This finishes the proof of Proposition 2.7. □

Deduction of Theorem 1.1 from Theorem 1.3. Applying Lemma 2.3 and Proposition 2.7 we get $[Z \oplus M, M] = [Z \oplus M, N]$. Then the variety $\overline{\mathcal{O}}_M$ is regular at the point N, by [11, Proposition 2.2].

3. Bimodules over $k[x, y]/(x^3 - y^2)$

Let $R = k[m^2, m^3]$ denote the subalgebra of the polynomial ring $k[m]$ in a formal variable m. We say that a left R-module M has property [P1] if the sequence

$$\begin{pmatrix} M \\ M \end{pmatrix} \xrightarrow{\begin{pmatrix} m^3 & -m^2 \\ m^4 & -m^3 \end{pmatrix}\cdot} \begin{pmatrix} M \\ M \end{pmatrix} \xrightarrow{\begin{pmatrix} m^3 & -m^2 \\ m^4 & -m^3 \end{pmatrix}\cdot} \begin{pmatrix} M \\ M \end{pmatrix}$$

is exact. We shall see later (Corollary 5.3) that this is equivalent to the fact that M has a free resolution of finite length. Dually, we say that a right R-module M has property [P1′] if the sequence

$$\begin{pmatrix} M & M \end{pmatrix} \xrightarrow{\cdot\begin{pmatrix} m^3 & m^4 \\ -m^2 & -m^3 \end{pmatrix}} \begin{pmatrix} M & M \end{pmatrix} \xrightarrow{\cdot\begin{pmatrix} m^3 & m^4 \\ -m^2 & -m^3 \end{pmatrix}} \begin{pmatrix} M & M \end{pmatrix}$$

is exact. Observe that $\mathfrak{m} = (m^2, m^3)$ is a maximal ideal of R.

Lemma 3.1. *Let M be a submodule of a left free R-module. If M has property* [P1] *then it is free.*

Proof. Let $\{b_s\}_{s\in S}$ be a set of elements of M whose residue classes form a linear basis of $M/\mathfrak{m}M$. We want to show that this set is a basis of the R-module M. Since M is contained in a free R-module W then

$$\bigcap_{i\geqslant 1}\mathfrak{m}^i M\subseteq\bigcap_{i\geqslant 1}\mathfrak{m}^i W=\{0\}.$$

By Nakayama's lemma, the elements b_s, $s\in S$ generate the R-module M.

Assume that $\sum_{s\in S} r_s b_s=0$, where all but a finite number of elements $r_s\in R$ are zero. We decompose $r_s=a_{s,0}+\sum_{i\geqslant 2} a_{s,i}m^i$, $s\in S$, where $a_{s,i}$ are scalars in k. It follows from the definition of b_s, $s\in S$ that $a_{s,0}=0$ for any $s\in S$. We have to show that $r_s=0$ for any $s\in S$, which means that $a_{s,i}=0$ for any $s\in S$ and $i\geqslant 2$. Suppose this is not the case and let $j\geqslant 2$ denote the minimal integer such that there is some $s_0\in S$ with $a_{s_0,j}\neq 0$. Then

$$0=m^3\left(\sum_{s\in S}\left(\sum_{i\geqslant j} a_{s,i}m^i\right)b_s\right)=m^j\left(\sum_{s\in S}\left(\sum_{i\geqslant 3} a_{s,j+i-3}m^i\right)b_s\right).$$

Since M is contained in a free R-module, m^j is not a zero divisor in M. Consequently,

$$\sum_{s\in S}\left(\sum_{i\geqslant 3} a_{s,j+i-3}m^i\right)b_s=0.$$

Then $m^3x'-m^2x''=0$ for

$$x'=\sum_{s\in S} a_{s,j}b_s\quad\text{and}\quad x''=-\sum_{s\in S}\left(\sum_{i\geqslant 2} a_{s,j+i-1}m^i\right)b_s.$$

Moreover,

$$0=m^3\big(m^3x'-m^2x''\big)=m^2\big(m^4x'-m^3x''\big).$$

Since m^2 is not a zero divisor in M and M has property [P1], we get

$$\begin{pmatrix} m^3 & -m^2\\ m^4 & -m^3\end{pmatrix}\begin{pmatrix} x'\\ x''\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix} m^3 & -m^2\\ m^4 & -m^3\end{pmatrix}\begin{pmatrix} y'\\ y''\end{pmatrix}=\begin{pmatrix} x'\\ x''\end{pmatrix}$$

for some $y',y''\in M$. Therefore x' belongs to $\mathfrak{m}M$, and consequently, $a_{s,j}=0$ for any $s\in S$. This gives a contradiction with the choice of j and hence the module M is free. $\square$

Let M be an R-R-bimodule, $N = \left(\begin{smallmatrix} M & M \\ M & M \end{smallmatrix}\right)$ and consider the maps

$$\xi : N \xrightarrow{\left(\begin{smallmatrix} m^3 & -m^2 \\ m^4 & -m^3 \end{smallmatrix}\right)\cdot} N \quad \text{and} \quad \eta : N \xrightarrow{\cdot\left(\begin{smallmatrix} m^3 & m^4 \\ -m^2 & -m^3 \end{smallmatrix}\right)} N,$$

given by left and right, respectively, multiplications of N by (2×2)-matrices. Observe that $\xi\xi = \eta\eta = 0$ and $\xi\eta = \eta\xi$. We say that M has property [P2] if the sequence

$$N \xrightarrow{\xi\eta} N \xrightarrow{\left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right)} N \oplus N$$

is exact. In fact, we shall see (Corollary 5.2) that property [P2] is equivalent to the fact that the bimodule M has a free resolution of finite length.

Lemma 3.2. *Let $N = \left(\begin{smallmatrix} M & M \\ M & M \end{smallmatrix}\right)$, where M is an R-R-bimodule having property* [P2]. *Then M has properties* [P1] *and* [P1′], *and the following sequence is exact*:

$$N \oplus N \xrightarrow{(\xi\ \eta)} N \xrightarrow{\xi\eta} N \xrightarrow{\left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right)} N \oplus N \xrightarrow{\left(\begin{smallmatrix} \xi & 0 \\ \eta & -\xi \\ 0 & \eta \end{smallmatrix}\right)} N \oplus N \oplus N. \tag{3.1}$$

Proof. Observe that M has property [P1] if and only if the sequence

$$N \xrightarrow{\xi} N \xrightarrow{\xi} N$$

is exact. We take $n \in N$ such that $\xi(n) = 0$ and set $n_1 = \eta(n)$. Then $\xi(n_1) = \eta(n_1) = 0$, which implies that $n_1 = \eta\xi(n_2)$ for some $n_2 \in N$. Let $n_3 = n - \xi(n_2)$. Then $\xi(n_3) = \eta(n_3) = 0$, which gives $n_3 = \xi\eta(n_4)$ for some $n_4 \in N$. Consequently, $n = \xi(n_2 + \eta(n_4))$. This shows that the sequence $N \xrightarrow{\xi} N \xrightarrow{\xi} N$ is exact. By a similar diagram chasing, one can get the exactness of the sequences $N \xrightarrow{\eta} N \xrightarrow{\eta} N$ and (3.1), which proves the claim. □

Lemma 3.3. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-R-bimodules. If two of the bimodules M_1, M_2 and M_3 have property* [P2], *then the third one has as well.*

Proof. Assume that $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of R-R-bimodules. Then we get the exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$, where

$$N_i = \begin{pmatrix} M_i & M_i \\ M_i & M_i \end{pmatrix} \qquad \text{for } i = 1, 2, 3.$$

We apply Lemma 3.2 and consider the commutative diagram with exact rows

$$\begin{tikzcd}
 & & (N_2)^2 \arrow[r] \arrow[d, "(\xi\ \eta)"] & (N_3)^2 \arrow[r] \arrow[d, "(\xi\ \eta)"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d, "\xi\eta"] & N_2 \arrow[r] \arrow[d, "\xi\eta"] & N_3 \arrow[r] \arrow[d, "\xi\eta"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d, "\binom{\xi}{\eta}"] & N_2 \arrow[r] \arrow[d, "\binom{\xi}{\eta}"] & N_3 \arrow[r] \arrow[d, "\binom{\xi}{\eta}"] & 0 \\
0 \arrow[r] & (N_1)^2 \arrow[r] \arrow[d, "\begin{pmatrix} \xi & 0 \\ \eta & -\xi \\ 0 & \eta \end{pmatrix}"] & (N_2)^2 \arrow[r] \arrow[d, "\begin{pmatrix} \xi & 0 \\ \eta & -\xi \\ 0 & \eta \end{pmatrix}"] & (N_3)^2 \arrow[r] & 0 \\
0 \arrow[r] & (N_1)^3 \arrow[r] & (N_2)^3. & &
\end{tikzcd}$$

If two of the bimodules M_1, M_2 and M_3 have property [P2] then the corresponding two columns are exact. Hence we get the exactness in the middle of the third column, which means that the third bimodule has also property [P2]. □

Lemma 3.4. *Any free R-R-bimodule has property* [P2].

Proof. Let M be a free R-R-bimodule and choose a basis $\{b_s\}_{s\in S}$. Assume that $x_{1,1}$, $x_{1,2}$, $x_{2,1}$ and $x_{2,2}$ are elements in M such that

$$\begin{pmatrix} m^3 & -m^2 \\ m^4 & -m^3 \end{pmatrix}\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}\begin{pmatrix} m^3 & m^4 \\ -m^2 & -m^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

We decompose

$$x_{p,q} = \sum_{s\in S} \sum_{\substack{i\geqslant 0 \\ i\neq 1}} \sum_{\substack{j\geqslant 0 \\ j\neq 1}} a^{p,q}_{s,i,j} m^i b_s m^j, \quad p, q = 1, 2,$$

where all but a finite number of scalars $a^{p,q}_{s,i,j}$ in k are zero. We conclude from (3.2) that

$$a^{1,1}_{s,i,j} = a^{1,2}_{s,i,j+1} = a^{2,1}_{s,i+1,j} = a^{2,2}_{s,i+1,j+1} \quad \text{for } s \in S,\ i, j \geqslant 2,$$

and the remaining scalars $a^{p,q}_{s,i,j}$ are zero. If we set

$$y_{1,1} = \sum_{s\in S} a^{1,1}_{s,3,3} b_s, \qquad y_{1,2} = -\sum_{s\in S} \sum_{\substack{j\geqslant 0 \\ j\neq 1}} a^{1,1}_{s,3,j+2} b_s m^j,$$

$$y_{2,1} = -\sum_{s\in S}\sum_{\substack{i\geqslant 0\\ i\neq 1}} a^{1,1}_{s,i+2,3} m^i b_s, \qquad y_{2,2} = \sum_{s\in S}\sum_{\substack{i\geqslant 0\\ i\neq 1}}\sum_{\substack{j\geqslant 0\\ j\neq 1}} a^{1,1}_{s,i+2,j+2} m^i b_s m^j,$$

then

$$\begin{pmatrix} x_{1,1} & x_{1,2}\\ x_{2,1} & x_{2,2}\end{pmatrix} = \begin{pmatrix} m^3 & -m^2\\ m^4 & -m^3\end{pmatrix}\begin{pmatrix} y_{1,1} & y_{1,2}\\ y_{2,1} & y_{2,2}\end{pmatrix}\begin{pmatrix} m^3 & m^4\\ -m^2 & -m^3\end{pmatrix}.$$

Hence M has property [P2]. $\square$

Lemma 3.5. *Assume that M is an R-R-bimodule having property* [P2] *which is torsion free as a right R-module. Then the left R-module $M/M\mathfrak{m}$ has property* [P1] *and the following sequence is exact*:

$$M/Mm^2 \xrightarrow{\cdot m^3} M/Mm^2 \xrightarrow{\cdot m^3} M/Mm^2. \tag{3.3}$$

Proof. Since m^2 is not a zero divisor of the right R-module M, then the sequence

$$0 \to M \xrightarrow{\cdot m^2} M \to M/Mm^2 \to 0$$

is exact. In fact, this is a sequence of R-R-bimodules since the algebra R is commutative. Then M/Mm^2 has property [P2], by Lemma 3.3. Applying Lemma 3.2 we get the exact sequence

$$N \oplus N \xrightarrow{(\xi\ \eta)} N \xrightarrow{\xi\eta} N, \quad \text{where } N = \begin{pmatrix} M/Mm^2 & M/Mm^2\\ M/Mm^2 & M/Mm^2\end{pmatrix}.$$

We have to show the exactness of the sequence

$$\begin{pmatrix} M/M\mathfrak{m}\\ M/M\mathfrak{m}\end{pmatrix} \xrightarrow{\begin{pmatrix} m^3 & -m^2\\ m^4 & -m^3\end{pmatrix}\cdot} \begin{pmatrix} M/M\mathfrak{m}\\ M/M\mathfrak{m}\end{pmatrix} \xrightarrow{\begin{pmatrix} m^3 & -m^2\\ m^4 & -m^3\end{pmatrix}\cdot} \begin{pmatrix} M/M\mathfrak{m}\\ M/M\mathfrak{m}\end{pmatrix}.$$

Let x_1 and x_2 be elements in M such that

$$\begin{pmatrix} m^3 & -m^2\\ m^4 & -m^3\end{pmatrix}\begin{pmatrix} x_1 + M\mathfrak{m}\\ x_2 + M\mathfrak{m}\end{pmatrix} = \begin{pmatrix} 0 + M\mathfrak{m}\\ 0 + M\mathfrak{m}\end{pmatrix}.$$

Then

$$\begin{pmatrix} m^3 & -m^2\\ m^4 & -m^3\end{pmatrix}\begin{pmatrix} x_1 & 0\\ x_2 & 0\end{pmatrix}\begin{pmatrix} m^3 & m^4\\ -m^2 & -m^3\end{pmatrix} \in \begin{pmatrix} M\mathfrak{m} & 0\\ M\mathfrak{m} & 0\end{pmatrix}\begin{pmatrix} m^3 & m^4\\ -m^2 & -m^3\end{pmatrix} \subseteq \begin{pmatrix} Mm^2 & Mm^2\\ Mm^2 & Mm^2\end{pmatrix},$$

hence

$$\xi\eta\begin{pmatrix} x_1 + Mm^2 & 0 + Mm^2\\ x_2 + Mm^2 & 0 + Mm^2\end{pmatrix} = \begin{pmatrix} 0 + Mm^2 & 0 + Mm^2\\ 0 + Mm^2 & 0 + Mm^2\end{pmatrix},$$

and consequently,

$$\begin{pmatrix} x_1 + Mm^2 & 0 + Mm^2 \\ x_2 + Mm^2 & 0 + Mm^2 \end{pmatrix} = \begin{pmatrix} m^3 & -m^2 \\ m^4 & -m^3 \end{pmatrix} \begin{pmatrix} y_1 + Mm^2 & y_2 + Mm^2 \\ y_3 + Mm^2 & y_4 + Mm^2 \end{pmatrix} + \begin{pmatrix} y_5 + Mm^2 & y_6 + Mm^2 \\ y_7 + Mm^2 & y_8 + Mm^2 \end{pmatrix} \begin{pmatrix} m^3 & m^4 \\ -m^2 & -m^3 \end{pmatrix}$$

for some $y_1, \dots, y_8 \in M$. This implies that

$$\begin{pmatrix} x_1 + M\mathfrak{m} \\ x_2 + M\mathfrak{m} \end{pmatrix} = \begin{pmatrix} m^3 & -m^2 \\ m^4 & -m^3 \end{pmatrix} \begin{pmatrix} y_1 + M\mathfrak{m} \\ y_3 + M\mathfrak{m} \end{pmatrix}.$$

Therefore $M/M\mathfrak{m}$ has property [P1]. We know that M/Mm^2 has property [P1′], by Lemma 3.2. This gives the exact sequence

$$\begin{pmatrix} M/Mm^2 & M/Mm^2 \end{pmatrix} \xrightarrow{\cdot \begin{pmatrix} m^3 & 0 \\ 0 & -m^3 \end{pmatrix}} \begin{pmatrix} M/Mm^2 & M/Mm^2 \end{pmatrix} \xrightarrow{\cdot \begin{pmatrix} m^3 & 0 \\ 0 & -m^3 \end{pmatrix}} \begin{pmatrix} M/Mm^2 & M/Mm^2 \end{pmatrix},$$

from which we derive the exactness of (3.3). □

Proposition 3.6. *Let M be an R-R-bimodule having property* [P2]. *If M is torsion free as a right R-module, then there is a free bimodule resolution*

$$0 \to U \to W \to M \to 0.$$

Furthermore, if the bimodule M is finitely generated then we may assume the same for U and W.

Proof. We take an exact sequence of R-R-bimodules

$$0 \to U \to W \to M \to 0$$

such that the bimodule W is free. Obviously W can be finitely generated provided M is finitely generated. Since the R-R-bimodules can be equivalently considered as $R \otimes R$-modules and the ring $R \otimes R$ is noetherian, then the bimodule U is finitely generated if W

is. Furthermore, U, W and M are torsion free right R-modules. Hence we get the following commutative diagram with exact columns and upper two rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U \arrow[r] \arrow[d, "\cdot m^2"] & W \arrow[r] \arrow[d, "\cdot m^2"] & M \arrow[r] \arrow[d, "\cdot m^2"] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U/Um^2 \arrow[r] \arrow[d] & W/Wm^2 \arrow[r] \arrow[d] & M/Mm^2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Consequently, the bottom row is also exact. Applying lemma (3.3) we get another commutative diagram with exact columns and upper three rows

$$\begin{tikzcd}
 & & W/Wm^2 \arrow[r] \arrow[d, "\cdot m^3"] & M/Mm^2 \arrow[r] \arrow[d, "\cdot m^3"] & 0 \\
0 \arrow[r] & U/Um^2 \arrow[r] \arrow[d, "\cdot m^3"] & W/Wm^2 \arrow[r] \arrow[d, "\cdot m^3"] & M/Mm^2 \arrow[r] \arrow[d, "\cdot m^3"] & 0 \\
0 \arrow[r] & U/Um^2 \arrow[r] \arrow[d] & W/Wm^2 \arrow[r] \arrow[d] & M/Mm^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U/U\mathfrak{m} \arrow[r] \arrow[d] & W/W\mathfrak{m} \arrow[r] \arrow[d] & M/M\mathfrak{m} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Hence the bottom row is also exact. The bimodule U has the property [P2], by Lemmas 3.3 and 3.4. Then $U/U\mathfrak{m}$ has property [P1], by Lemma 3.5. Since $W/W\mathfrak{m}$ is a free left R-module then $U/U\mathfrak{m}$ is also free, by Lemma 3.1. Let $\{b_s\}_{s\in S}$ be a set of elements of U whose residue classes form a basis of the free left R-module $U/U\mathfrak{m}$. We want to show that this set is a basis of the R-R-bimodule U. Since

$$\bigcap_{i\geqslant 1} U\mathfrak{m}^i \subseteq \bigcap_{i\geqslant 1} W\mathfrak{m}^i = \{0\},$$

then the elements b_s, $s \in S$ generate the bimodule U, by Nakayama's lemma. Assume that

$$\sum_{s\in S}\left(r_{s,0}b_s + \sum_{i\geqslant 2} r_{s,i} b_s m^i\right) = 0,$$

where all but a finite number of elements $r_{s,i} \in R$ are zero. It follows from the definition of b_s, $s \in S$ that $r_{s,0} = 0$ for any $s \in S$. Repeating arguments as in the proof of Lemma 3.1 and using the fact that U has property [P1′], by Lemma 3.2, we get that $r_{s,i} = 0$ for any $s \in S$ and $i \geqslant 2$. Hence the bimodule U is free. $\square$

4. The proof of Theorem 1.3

Suppose that the exact sequence in mod A

$$0 \to Z \xrightarrow{\binom{\tilde{f}}{\tilde{g}}} Z \oplus Y \xrightarrow{(\tilde{f}, -\tilde{h})} Z \to 0 \tag{4.1}$$

with Z indecomposable does not split and that $[Z, Z] - [Y, Y] = 1$. Then $\tilde{f}$ is nilpotent and Y is not isomorphic to Z. Furthermore,

$$[Y, Y] = [Y, Z] = [Z, Y] = [Z, Z] - 1,$$

by Lemmas 2.1 and 2.2 applied to the sequence (4.1). This leads to the following exact sequences induced by (4.1):

$$0 \to \operatorname{Hom}_A(Y, Z) \to \operatorname{Hom}_A(Y, Z \oplus Y) \to \operatorname{Hom}(Y, Z) \to 0, \tag{4.2}$$

$$0 \to \operatorname{Hom}_A(Z, Z) \to \operatorname{Hom}_A(Z, Z \oplus Y) \to \operatorname{rad}(Z, Z) \to 0, \tag{4.3}$$

$$0 \to \operatorname{Hom}_A(Z, Y) \to \operatorname{Hom}_A(Z \oplus Y, Y) \to \operatorname{Hom}(Z, Y) \to 0, \tag{4.4}$$

$$0 \to \operatorname{Hom}_A(Z, Z) \to \operatorname{Hom}_A(Z \oplus Y, Z) \to \operatorname{rad}(Z, Z) \to 0. \tag{4.5}$$

Let Q be the quiver

$$\begin{tikzcd} y \arrow[r, bend left, "h"] & z \arrow[l, bend left, "g"] \arrow[loop right, "f"] \end{tikzcd}$$

and $\Lambda = kQ/(f^2 - hg)$ be the quotient of the path algebra of Q by the two-sided ideal generated by $f^2 - hg$. We denote by ε_y and ε_z the idempotents corresponding to the vertices y and z, respectively. In particular, $1_\Lambda = \varepsilon_y + \varepsilon_z$. It is easy to see that

$$\mathcal{B} = \{\varepsilon_y, \varepsilon_z, f^{i+1}, gf^i, f^i h, gf^i h;\ i \geqslant 0\}$$

is a multiplicative basis of Λ, that is, $\mathcal{B}$ is a basis of the underlying vector space of Λ such that $b_1 b_2$ belongs to $\mathcal{B}$ or equals zero, for any b_1 and b_2 in $\mathcal{B}$.

Since $(\tilde{f})^2 = \tilde{h}\tilde{g}$, we have a canonical algebra homomorphism

$$\Phi : \Lambda \to \mathrm{End}_A(Y \oplus Z),$$

$$\varepsilon_y \mapsto \begin{pmatrix} 1_Y & 0 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon_z \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1_Z \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 0 & \tilde{f} \end{pmatrix}, \quad g \mapsto \begin{pmatrix} 0 & \tilde{g} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad h \mapsto \begin{pmatrix} 0 & 0 \\ \tilde{h} & 0 \end{pmatrix}.$$

This allows us to consider the algebra $E = \mathrm{End}_A(Y \oplus Z)$ as a Λ-Λ-bimodule via

$$\lambda_1 \cdot e \cdot \lambda_2 = \Phi(\lambda_1) e \Phi(\lambda_2),$$

for any $\lambda_1, \lambda_2 \in \Lambda$ and $e \in \mathrm{End}_A(Y \oplus Z)$. In particular, $\mathrm{End}_A(Y) = \varepsilon_y E \varepsilon_y$.

Let $E' = \begin{pmatrix} \mathrm{End}_A(Y) & \mathrm{Hom}_A(Z,Y) \\ \mathrm{Hom}_A(Y,Z) & \mathrm{rad}(Z,Z) \end{pmatrix}$. Then E' is a subbimodule in E. We derive from a direct sum of (4.2) and (4.3) the exact sequence

$$0 \to \varepsilon_z E \xrightarrow{\binom{f}{g}\cdot} \varepsilon_z E \oplus \varepsilon_y E \xrightarrow{(f,-h)\cdot} \varepsilon_z E' \to 0. \tag{4.6}$$

We denote by R the algebra $\varepsilon_y \Lambda \varepsilon_y$ with $1_R = \varepsilon_y$. The set $\{\varepsilon_y, gf^i h;\ i \geqslant 0\}$ is a multiplicative basis of R. Since $(gf^i h) \cdot (gf^j h) = gf^{i+j+2} h$, the algebra R is commutative and it will be convenient to identify R as the subalgebra $k[m^2, m^3]$ of the polynomial algebra $k[m]$, where $gf^i h = m^{i+2}$ for any $i \geqslant 0$. Then $\mathrm{End}_A(Y)$ is an R-R-bimodule. Let $\{b_1, \ldots, b_s\}$ be a set of generators of the bimodule $\mathrm{End}_A(Y)$ (for instance we may take a basis of the finite dimensional vector space $\mathrm{End}_A(Y)$). Let $\Omega = \Lambda \oplus \bigoplus_{i=1}^s (\Lambda \varepsilon_y \otimes \varepsilon_y \Lambda)$. Since $\Lambda \varepsilon_y \otimes \varepsilon_y \Lambda$ is a projective Λ-Λ-bimodule, we may define the bimodule homomorphism $\Psi : \Omega \to E$,

$$\Psi\bigl(\lambda, \lambda_1 \varepsilon_y \otimes \varepsilon_y \lambda_1', \ldots, \lambda_s \varepsilon_y \otimes \varepsilon_y \lambda_s'\bigr) = \lambda \cdot 1_E + \sum_{i=1}^{s} \lambda_i \cdot \begin{pmatrix} b_i & 0 \\ 0 & 0 \end{pmatrix} \cdot \lambda_i'.$$

Lemma 4.1. *The homomorphism Ψ is surjective.*

Proof. Since $b_1, \ldots, b_s$ are generators of $\mathrm{End}_A(Y)$, the latter is contained in $\mathrm{Im}(\Psi)$. We derive from (4.2) the exact squares

$$\begin{tikzcd}
\mathrm{Hom}_A(Y,Z) \arrow[r, "\mathrm{Hom}_A(Y,\tilde{f})"] \arrow[d, "\mathrm{Hom}_A(Y,\tilde{g})"'] & \mathrm{Hom}_A(Y,Z) \arrow[r, "\mathrm{Hom}_A(Y,\tilde{g})"] \arrow[d, "\mathrm{Hom}_A(Y,\tilde{f})"'] & \mathrm{End}_A(Y) \arrow[d, "\mathrm{Hom}_A(Y,\tilde{h})"] \\
\mathrm{End}_A(Y) \arrow[r, "\mathrm{Hom}_A(Y,\tilde{h})"] & \mathrm{Hom}_A(Y,Z) \arrow[r, "\mathrm{Hom}_A(Y,\tilde{f})"] & \mathrm{Hom}_A(Y,Z).
\end{tikzcd}$$

Consequently, $\mathrm{Hom}_A(Y,Z) = fh \cdot \mathrm{End}_A(Y) + h \cdot \mathrm{End}_A(Y)$ is contained in the Λ-Λ-bimodule $\mathrm{Im}(\Psi)$. Similarly, $\mathrm{Im}(\Psi)$ contains

$$\mathrm{Hom}_A(Z,Y) = \mathrm{End}_A(Y) \cdot g + \mathrm{End}_A(Y) \cdot gf.$$

Let $e \in \operatorname{End}_A(Z)$. It follows from (4.3) that

$$e = \mu_1 \cdot 1_Z + \tilde{f}e' + \tilde{h}d \quad \text{and} \quad e' = \mu_2 \cdot 1_Z + \tilde{f}e'' + \tilde{h}d'$$

for some scalars $\mu_1, \mu_2 \in k$ and A-homomorphisms d, d', e' and e''. Hence

$$e = \mu_1 \cdot 1_Z + \mu_2 \cdot \tilde{f} + \tilde{h}\tilde{g}e'' + \tilde{f}\tilde{h}d' + \tilde{h}d$$

belongs to

$$\Psi(\mu_1 \cdot \varepsilon_z + \mu_2 \cdot f, 0, \dots, 0) + h \cdot \operatorname{Hom}_A(Z, Y) + fh \cdot \operatorname{Hom}_A(Z, Y).$$

Therefore $\operatorname{Im}(\Psi)$ contains $\operatorname{End}_A(Z)$ as well. $\square$

Let Λ' denote the subspace of Λ generated by $\mathcal{B} \setminus \{\varepsilon_z\}$. Furthermore, let $\Omega' = \Lambda' \oplus \bigoplus_{i=1}^{s}(\Lambda\varepsilon_y \otimes \varepsilon_y\Lambda)$. It is easy to see that Λ' is a two-sided ideal of Λ and Ω' is a Λ-Λ-subbimodule of Ω.

Lemma 4.2. *The following sequence is exact*:

$$0 \to \varepsilon_z\Omega \xrightarrow{\binom{f}{g}\cdot} \varepsilon_z\Omega \oplus \varepsilon_y\Omega \xrightarrow{(f,-h)\cdot} \varepsilon_z\Omega' \to 0. \tag{4.7}$$

Proof. $\mathcal{B}$ induces canonically a basis $\mathcal{C}$ of the bimodule Ω such that the set $\mathcal{C} \cup \{0\}$ is invariant under left and right multiplications by f, g and h. Furthermore, suitable subsets of $\mathcal{C}$ give bases of the spaces $\varepsilon_y\Omega$, $\varepsilon_z\Omega$, $\varepsilon_z\Omega'$, $\Omega\varepsilon_y$, $\Omega\varepsilon_z$ and $\Omega'\varepsilon_z$. Now straightforward calculations on these bases are left to the reader. $\square$

Let J denote the kernel of $\Psi : \Omega \to \operatorname{End}_A(Y \oplus Z)$.

Lemma 4.3. *The following sequences are exact*:

$$0 \to \varepsilon_z J \xrightarrow{\binom{g}{gf}\cdot} \varepsilon_y J \oplus \varepsilon_y J \xrightarrow{(fh,-h)\cdot} \varepsilon_z J \to 0, \tag{4.8}$$

$$0 \to J\varepsilon_z \xrightarrow{\cdot\binom{h}{fh}} J\varepsilon_y \oplus J\varepsilon_y \xrightarrow{\cdot(gf,-g)} J\varepsilon_z \to 0. \tag{4.9}$$

Proof. Observe that $\Psi(\Omega') \subseteq E'$ and let e denote the element $(\varepsilon_z, 0, \dots, 0)$ in Ω. It follows from the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Omega' \arrow[r] \arrow[d, "\Psi'"'] & \Omega \arrow[r] \arrow[d, "\Psi"] & k \cdot \bar{e} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E' \arrow[r] & E \arrow[r] & k \cdot \overline{1_Z} \arrow[r] & 0
\end{tikzcd}$$

that J is also the kernel of the restriction $\Psi' : \Omega' \to E'$ of Ψ. Applying (4.6) and (4.7) we get the following commutative diagram with exact columns and exact two bottom rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \varepsilon_z J \arrow[rr, "\binom{f}{g}\cdot"] \arrow[d] & & \varepsilon_z J \oplus \varepsilon_y J \arrow[rr, "(f,-h)\cdot"] \arrow[d] & & \varepsilon_z J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varepsilon_z \Omega \arrow[rr, "\binom{f}{g}\cdot"] \arrow[d] & & \varepsilon_z \Omega \oplus \varepsilon_y \Omega \arrow[rr, "(f,-h)\cdot"] \arrow[d] & & \varepsilon_z \Omega' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varepsilon_z E \arrow[rr, "\binom{f}{g}\cdot"] \arrow[d] & & \varepsilon_z E \oplus \varepsilon_y E \arrow[rr, "(f,-h)\cdot"] \arrow[d] & & \varepsilon_z E' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & & 0. &
\end{tikzcd}$$

Hence the upper row is also exact. Now the exactness of (4.8) follows from joining the exact squares

$$\begin{tikzcd}
\varepsilon_z J \arrow[r, "f\cdot"] \arrow[d, "g\cdot"'] & \varepsilon_z J \arrow[r, "g\cdot"] \arrow[d, "f\cdot"] & \varepsilon_y J \arrow[d, "h\cdot"] \\
\varepsilon_y J \arrow[r, "h\cdot"] & \varepsilon_z J \arrow[r, "f\cdot"] & \varepsilon_z J.
\end{tikzcd}$$

By duality, the sequence (4.9) is also exact. $\square$

Observe that $I = \varepsilon_y J \varepsilon_y$ is an R-R-subbimodule of $\varepsilon_y \Omega \varepsilon_y = R \oplus \bigoplus_{i=1}^{s} (R \otimes R)$. Hence we get the exact sequence of R-R-bimodules

$$0 \to I \to R \oplus \bigoplus_{i=1}^{s} (R \otimes R) \to \mathrm{End}_A(Y) \to 0. \tag{4.10}$$

In particular, I is torsion free as a right R-module. Obviously the bimodule $R \oplus \bigoplus_{i=1}^{s} (R \otimes R)$ is generated by the elements $e_0 = (1, 0, \dots, 0)$ and

$$e_i = (0, \dots, 0, 1 \otimes 1, 0, \dots, 0), \quad i = 1, \dots, s.$$

Furthermore, the bimodule I is finitely generated, since it is a subbimodule of a finitely generated R-R-bimodule and the ring $R \otimes R$ is noetherian.

Lemma 4.4. *The R-R-bimodule I has property* [P2].

Proof. Applying Lemma 4.3 we get the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (\varepsilon_z J \varepsilon_z) \arrow[r, "\cdot(h\ fh)"] \arrow[d, "\binom{g}{gf}\cdot"'] & (\varepsilon_z J \varepsilon_y \quad \varepsilon_z J \varepsilon_y) \arrow[r, "\cdot\binom{gf}{-g}"] \arrow[d, "\binom{g}{gf}\cdot"] & (\varepsilon_z J \varepsilon_z) \arrow[r] \arrow[d, "\binom{g}{gf}\cdot"] & 0 \\
0 \arrow[r] & \begin{pmatrix} \varepsilon_y J \varepsilon_z \\ \varepsilon_y J \varepsilon_z \end{pmatrix} \arrow[r, "\cdot(h\ fh)"] \arrow[d, "(fh\ -h)\cdot"'] & \begin{pmatrix} I & I \\ I & I \end{pmatrix} \arrow[r, "\cdot\binom{gf}{-g}"] \arrow[d, "(fh\ -h)\cdot"] & \begin{pmatrix} \varepsilon_y J \varepsilon_z \\ \varepsilon_y J \varepsilon_z \end{pmatrix} \arrow[r] \arrow[d, "(fh\ -h)\cdot"] & 0 \\
0 \arrow[r] & (\varepsilon_z J \varepsilon_z) \arrow[r, "\cdot(h\ fh)"] \arrow[d] & (\varepsilon_z J \varepsilon_y \quad \varepsilon_z J \varepsilon_y) \arrow[r, "\cdot\binom{gf}{-g}"] \arrow[d] & (\varepsilon_z J \varepsilon_z) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now the claim follows from the fact that any commutative diagram in a module category

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & C_1 \arrow[r, "\beta_1"] \arrow[d, "\gamma_2"] & B \arrow[r] \arrow[d, "\alpha_2"] & 0 \\
0 \arrow[r] & C_2 \arrow[r, "\gamma_1"] \arrow[d, "\beta_2"'] & D \arrow[r, "\delta_1"] \arrow[d, "\delta_2"] & C_2 \arrow[r] \arrow[d, "\beta_2"] & 0 \\
0 \arrow[r] & B \arrow[r, "\alpha_1"] \arrow[d] & C_1 \arrow[r, "\beta_1"] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns induces the exact sequence

$$D \xrightarrow{\gamma_2\alpha_1\beta_2\delta_1} D \xrightarrow{\binom{\gamma_1\delta_1}{\gamma_2\delta_2}} D \oplus D. \qquad \square$$

Applying Proposition 3.6 we get an exact sequence of finitely generated R-R-bimodules

$$0 \to U \to W \to I \to 0,$$

where the bimodules U and W are free. Let $\{u_1, \ldots, u_p\}$ be a basis of the bimodule U and $\{w_1, \ldots, w_q\}$ be a basis of W. Since the endomorphism $\tilde{f}$ is nilpotent, then $\tilde{g}(\tilde{f})^{t-2}\tilde{h} = 0$ for some $t \geqslant 2$. Hence m^t is an annihilator of the left R-module $\mathrm{End}_A(Y)$. Consequently, I contains $m^t(R \oplus \bigoplus_{i=1}^{s}(R \otimes R))$, by (4.10). Let z_i be an element of W such that its image is equal to $m^t e_i$ for $i = 1, \ldots, s$.

From now on, we shall consider the R-R-bimodules as left modules over the algebra $R' = R \otimes R = k[m^2, m^3, n^2, n^3]$, where the right multiplications by m^2 and m^3 are replaced by the multiplications by n^2 and n^3, respectively. The algebra R' is contained in $k[m, n]$ and the field $k(m, n)$ of fractions of $k[m, n]$ is also the field of fraction of R'. The tensor product of the R'-module monomorphisms

$$\left(R \oplus \bigoplus_{i=1}^{s}(R \otimes R)\right) \xrightarrow{m^t \cdot} I \to \left(R \oplus \bigoplus_{i=1}^{s}(R \otimes R)\right)$$

by the flat R'-module $k(m, n)$ leads to the monomorphisms

$$k(m, n)^s \to I \otimes_{R'} k(m, n) \to k(m, n)^s.$$

Since the composition is an isomorphism, each of the above maps is an isomorphism of vector spaces over the field $k(m, n)$. It follows from the exact sequence

$$0 \to U \otimes_{R'} k(m, n) \to W \otimes_{R'} k(m, n) \to I \otimes_{R'} k(m, n) \to 0$$

that $p + s = q$. Let D be the $(q \times q)$-matrix with coefficients in R' such that its rows represent the elements $u_1, \dots, u_p, z_1, \dots, z_s$ in the basis $w_1, \dots, w_q$, that is,

$$\begin{bmatrix} u_1 \\ \vdots \\ z_s \end{bmatrix} = D \begin{bmatrix} w_1 \\ \vdots \\ w_q \end{bmatrix}.$$

Lemma 4.5. $\det(D) = m^j$ *for some integer* j.

Proof. Let z_0 be an element of W whose image in I is equal to $m^t e_0$. Then $(m^2 - n^2)z_0$ belongs to U and hence

$$\left(m^2 - n^2\right)z_0 = r'_1 u_1 + \dots + r'_p u_p$$

for some elements r'_i in R'. This implies that

$$z_0 = \frac{1}{m^2 - n^2} \cdot v_2 \cdot \begin{bmatrix} u_1 \\ \vdots \\ z_s \end{bmatrix}, \qquad \text{where } v_2 = \left[r'_1, \dots, r'_p, 0, \dots, 0\right].$$

Since the image of $m^t w_i$ belongs to the R'-submodule in I generated by $m^t e_0, m^t e_1, \dots, m^t e_s$, then $m^t w_i$ belongs to the R'-submodule in W generated by $u_1, \dots, u_p, z_1, \dots, z_s$ and z_0, for $i = 1, \dots, q$. Consequently, there are a $(q \times q)$-matrix B with coefficients in R' and elements $r_1, \dots, r_q$ in R' such that

$$\begin{bmatrix} m^t w_1 \\ \vdots \\ m^t w_q \end{bmatrix} = B \begin{bmatrix} u_1 \\ \vdots \\ z_s \end{bmatrix} + \begin{bmatrix} r_1 \\ \vdots \\ r_q \end{bmatrix} \cdot z_0.$$

Observe that

$$\begin{bmatrix} m^t w_1 \\ \vdots \\ m^t w_q \end{bmatrix} = C \begin{bmatrix} u_1 \\ \vdots \\ z_s \end{bmatrix},$$

where

$$C = B + \frac{1}{m^2 - n^2} \cdot \begin{bmatrix} r_1 \\ \vdots \\ r_q \end{bmatrix} \cdot v_2$$

is a $(q \times q)$-matrix with coefficients in the field $k(m,n)$. Consequently, $C \cdot D = m^t \cdot I_q$, where I_q denotes the identity matrix. Hence

$$\det(C) \cdot \det(D) = m^{tq}.$$

Let D_i be the matrix obtained from B by replacing the ith row by v_2, for $i = 1, \dots, q$. Applying elementary properties of determinants we get

$$\det(C) = \det(B) + \frac{r_1}{m^2 - n^2} \det(D_1) + \cdots + \frac{r_q}{m^2 - n^2} \det(D_q).$$

Therefore $(m^2 - n^2) \cdot \det(C)$ is an element of $k[m^2, m^3, n^2, n^3]$ and $\det(D)$ is a divisor of $(m^2 - n^2) \cdot m^{tq}$ in the algebra $k[m^2, m^3, n^2, n^3]$. Replacing $(m^2 - n^2)$ by $(m^3 - n^3)$, we get that $\det(D)$ is also a divisor of $(m^3 - n^3) \cdot m^{tq}$. Since $k[m^2, m^3, n^2, n^3]$ is contained in the unique factorization domain $k[m,n]$ and the polynomials

$$\frac{m^2 - n^2}{m - n} = m + n \quad \text{and} \quad \frac{m^3 - n^3}{m - n} = m^2 + mn + n^2$$

are coprime, then $\det(D)$ is a divisor of $(m-n)m^{tq}$. Therefore $\det(D) = m^j$ for some integer j, as $(m-n)m^j$ does not belong to $k[m^2, m^3, n^2, n^3]$. $\square$

Applying Lemma 4.5 we get that the coefficients of the matrix $m^j \cdot D^{-1}$ belong to $k[m^2, m^3, n^2, n^3]$. Hence $m^j x$ belongs to the R'-submodule of W generated by $u_1, \dots, u_p, z_1, \dots, z_s$, for any $x \in W$. Therefore $m^j y$ belongs to the R'-submodule of I generated by $m^t e_1, \dots, m^t e_s$, for any $y \in I$. Taking $y = m^t \cdot e_0$ we obtain a contradiction. This finishes the proof of Theorem 1.3.

5. Corollaries and remarks

5.1. Theorem 1.1 can be generalized to other varieties. We give here two examples.

Let $Q = (Q_0, Q_1, s, e)$ be a finite quiver, where Q_0 is the set of vertices, Q_1 is the set of arrows and $s, e : Q_1 \to Q_0$ are functions such that any arrow α in Q_1 has the starting vertex $s(\alpha)$ and the ending vertex $e(\alpha)$. Let $\mathbf{d} = (d_i)_{i\in Q_0}$ be a sequence of positive integers. Furthermore, we denote by $\mathbb{M}_{d'\times d''}(k)$ the space of $(d' \times d'')$-matrices with coefficients in k, for any positive integers d' and d''. Then the group $\mathrm{GL}_{\mathbf{d}}(k) = \prod_{i\in Q_0} \mathrm{GL}_{d_i}(k)$ acts on the affine space $\mathrm{rep}_Q^{\mathbf{d}}(k) = \prod_{\alpha\in Q_1} \mathbb{M}_{d_{e(\alpha)}\times d_{s(\alpha)}}(k)$ by conjugations

$$(g_i)_{i\in Q_0} \star (m_\alpha)_{\alpha\in Q_1} = \left(g_{e(\alpha)} m_\alpha g_{s(\alpha)}^{-1}\right)_{\alpha\in Q_1}.$$

Let $d = \sum_{i\in Q_0} d_i$ and kQ denote the path algebra of Q. Then there is a fibre bundle

$$\mathcal{C}_{\mathbf{d}} \to \left(\mathrm{GL}_d(k)/\mathrm{GL}_{\mathbf{d}}(k)\right)$$

with typical fiber $\mathrm{rep}_Q^{\mathbf{d}}(k)$, where $\mathcal{C}_{\mathbf{d}}$ is a connected component of $\mathrm{mod}_{kQ}^d(k)$ (see [4]). Consequently, Theorem 1.1 remains true if we take the $\mathrm{GL}_{\mathbf{d}}(k)$-variety $\mathrm{rep}_Q^{\mathbf{d}}(k)$ instead of $\mathrm{mod}_A^d(k)$.

Let $P_1, \ldots, P_t$ be parabolic subgroups of $G = \mathrm{GL}_d(k)$. We consider the projective variety

$$X = G/P_1 \times \cdots \times G/P_t$$

equipped with the diagonal action of G. Applying arguments used in [6, §2] we get a G-equivariant principal H-bundle $\mathcal{U} \to X$, where $\mathrm{GL}_{\mathbf{d}}(k) = G \times H$ and $\mathcal{U}$ is a $\mathrm{GL}_{\mathbf{d}}(k)$-invariant open subset of $\mathrm{rep}_Q^{\mathbf{d}}(k)$, for some quiver Q and sequence $\mathbf{d}$. Thus Theorem 1.1 is still true if we replace the module variety by the G-variety X.

5.2. Let Q be the Kronecker quiver. Then the orbit closures in $\mathrm{rep}_Q^{\mathbf{d}}(k)$ are regular in codimension one, even if they contain infinitely many orbits [3]. It is an interesting question whether the orbit closures are regular in codimension one for the other extended Dynkin quivers.

5.3. We say that two exact sequences in $\mathrm{mod}\,A$

$$\sigma_l : 0 \to Z_l \xrightarrow{f_l} Z_l \oplus M \xrightarrow{g_l} N \to 0, \quad l = 1, 2,$$

with f_l in $\mathrm{rad}(Z_l, Z_l \oplus M)$ are equivalent if there is a commutative diagram in $\mathrm{mod}\,A$

$$\begin{tikzcd}
0 \arrow[r] & Z_1 \arrow[r, "f_1"] \arrow[d, "i"'] & Z_1 \oplus M \arrow[r, "g_1"] \arrow[d, "j"] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z_2 \arrow[r, "f_2"] & Z_2 \oplus M \arrow[r, "g_2"] & N \arrow[r] & 0
\end{tikzcd}$$

for some isomorphisms i, j. In particular, Z_1 is isomorphic to Z_2.

Corollary 5.1. *Let M and N be points in $\operatorname{mod}_A^d(k)$ such that N belongs to $\overline{\mathcal{O}}_M$ and $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 1$. Then there is a unique, up to an equivalence, exact sequence in $\operatorname{mod} A$*

$$0 \to Z \xrightarrow{f} Z \oplus M \xrightarrow{g} N \to 0 \tag{5.1}$$

with a radical morphism f. Furthermore, the module Z is indecomposable.

Proof. Applying Lemma 2.5 we get the exact sequence (5.1) with f radical and Z indecomposable. Let

$$\sigma' : 0 \to Z' \xrightarrow{f'} Z' \oplus M \xrightarrow{g'} N \to 0$$

be an exact sequence in $\operatorname{mod} A$ with f' in $\operatorname{rad}(Z', Z' \oplus M)$. By Lemma 2.3 and Proposition 2.7, $[Z \oplus M, M] = [Z \oplus M, N]$. Applying Lemma 2.1 for σ' and $X = Z \oplus M$ we get that g factors through g'. This leads to a commutative diagram in $\operatorname{mod} A$

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, "f"] \arrow[d, "i"'] & Z \oplus M \arrow[r, "g"] \arrow[d, "j"] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Z' \arrow[r, "f'"] & Z' \oplus M \arrow[r, "g'"] & N \arrow[r] & 0 \end{tikzcd}$$

for some homomorphisms i and j. We conclude from Lemma 2.2 that the induced exact sequence

$$0 \to Z \xrightarrow{\binom{f}{i}} (Z \oplus M) \oplus Z' \xrightarrow{(j, -f')} \big(Z' \oplus M\big) \to 0$$

splits. Hence i is a section and j is a retraction, as f and f' are radical homomorphisms. This implies that Z is a direct summand of Z' as well as $Z' \oplus M$ is a direct summand of $Z \oplus M$. Consequently, Z is isomorphic to Z', and i and j are isomorphisms. □

5.4. The bound for the difference of dimensions of $\operatorname{End}_A(Z)$ and $\operatorname{End}_A(Y)$ given in Theorem 1.3 is sharp. We recall here the example given in [1, 5.4]. Let $A = k[\alpha, \beta]/(\alpha^2, \beta^2)$ and Y and Z be modules in $\operatorname{mod}_A^4(k)$ such that

$$Y(\alpha) = Z(\alpha) = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix}, \qquad Y(\beta) = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{pmatrix}, \qquad Z(\beta) = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix},$$

and set

$$\tilde{f} = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}, \qquad \tilde{g} = \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{pmatrix}, \qquad \tilde{h} = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix}.$$

Then the module Z is indecomposable, the sequence

$$0 \to Z \xrightarrow{\binom{\tilde f}{\tilde g}} Z \oplus Y \xrightarrow{(\tilde f, -\tilde h)} Z \to 0$$

is exact and $\dim_k \mathrm{End}_A(Z) - \dim_k \mathrm{End}_A(Y) = 2$.

5.5. The properties [P1] and [P2] are strongly related to free resolutions of modules over $R = k[m^2, m^3]$ and $R' = k[m^2, m^3, n^2, n^3]$, respectively.

Corollary 5.2. *Let M be an R'-module. Then the following conditions are equivalent*:

(1) *M has property* [P2];
(2) *there is a free resolution*

$$0 \to F_2 \to F_1 \to F_0 \to M \to 0 \text{ of } M;$$

(3) *M has a free resolution of finite length.*

Proof. Obviously (2) implies (3). Furthermore, (3) implies (1), by Lemmas 3.3 and 3.4. Assume now that M has property [P2]. We take an exact sequence of R'-modules

$$0 \to M' \to F_0 \to M \to 0,$$

where the module F_0 is free. The module M' has property [P2], by Lemmas 3.3 and 3.4. Since M' is a torsion free bimodule, then (2) follows from Proposition 3.6. □

Corollary 5.3. *Let M be an R-module. Then the following conditions are equivalent*:

(1) *M has property* [P1];
(2) *there is a free resolution* $0 \to F_1 \to F_0 \to M \to 0$ *of* M;
(3) *M has a free resolution of finite length.*

Proof. The proof is similar to the previous one. We have to replace Proposition 3.6 by Lemma 3.1. Furthermore, one can repeat appropriate arguments to get versions of Lemmas 3.3 and 3.4 for R-modules. □

Observe that $N = \binom{M}{M}$ is a $k[\xi]/(\xi^2)$-module for any R-module M and $N' = \begin{pmatrix} M' & M' \\ M' & M' \end{pmatrix}$ is a $k[\xi, \eta]/(\xi^2, \eta^2)$-module for any R'-module M', where the residue classes of ξ and η denote the multiplications $\begin{pmatrix} m^3 & -m^2 \\ m^4 & -m^3 \end{pmatrix}\cdot$ and $\cdot\begin{pmatrix} n^3 & n^4 \\ -n^2 & -n^3 \end{pmatrix}$, respectively.

Since the algebras $k[\xi]/(\xi^2)$ and $k[\xi, \eta]/(\xi^2, \eta^2)$ are local and Frobenius, the free modules over them coincide with the projective modules and with the injective ones. One can prove that M has property [P1] if and only if N is a free $k[\xi]/(\xi^2)$-module, and M' has property [P2] if and only if N' is a free $k[\xi, \eta]/(\xi^2, \eta^2)$-module.

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